Toroidal Membrane under Internal Pressure

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In the case of a circular toroidal membrane under internal pressure there is no solution of the linear membrane equations for which the displacements are continuous. One possible resolution of the difficulty depends on introduction of an internal bending boundary layer. Another possibility more appropriate for very thin shells is to resort to the nonlinear membrane theory. The present paper is concerned with the second possibility. The equations are reduced to a linear second-order differential equation for the rotation of shell elements in which the coefficient of the undifferentiated term contains a large parameter. The equation is amenable to asymptotic methods of integration, and the solution is obtained in terms of two analytic functions of a single variable. These two functions, together with their first derivatives and certain integrals pertinent to the problem, are tabulated in the paper. With the help of these tables the title problem can be solved for practical ranges of the parameters without resort to further machine calculations.

Introduction

THE classical solution of the problem of the circular toroidal shell under internal pressure according to the linear membrane theory[1] gives reasonable stress distributions, but the displacements are discontinuous. The resolution of this difficulty depends on the use of a more accurate system of equations. For a certain range of parameters the linear bending theory is appropriate, and its use predicts the presence of an internal bending boundary layer necessary to insure continuity of displacements.[2, 3] However, for sufficiently thin shells where bending effects are in fact negligible, one must resort instead to the nonlinear membrane theory.

A solution of the problem according to the nonlinear membrane theory was given recently by Jordan.[4] The present paper is concerned with an alternative method of solution of the membrane equations with sufficient advantages over the method of Ref. 4 to justify re-examination of the problem. The question of determining the ranges of parameters in which the linear bending theory or the nonlinear membrane theory are valid has been answered by Reissner,[5] who finds an intermediate range where neither solution is valid. In Ref. 5 appropriate systems of equations for each of the three ranges are derived from a general nonlinear theory for shells of revolution. Solutions of the equations valid in the intermediate range are yet to be obtained.

Fundamental Equations

Several derivations of systems of equations appropriate for the analysis of a membrane of revolution in the nonlinear range are available in the literature.[5–8] The analysis in the present paper will be based upon an approximate system of equations given in Ref. 8, valid if strains are small and rotations are moderately small. These will be sufficiently accurate for the present purpose. In the case of a circular torus these are as follows:

Equilibrium

$$[rN_\phi\,(\sin\phi + \kappa\psi\cos\phi)]' = r(\cos\phi - \kappa\psi\sin\phi) \qquad (1)$$

$$[rN_\phi\,(\cos\phi - \kappa\psi\sin\phi)]' - N_\theta = -r(\sin\phi + \kappa\psi\cos\phi) \qquad (2)$$

Strain-Displacement

$$\epsilon_\phi = u'\cos\phi - w'\sin\phi + \tfrac{1}{2}\kappa\psi^2 \qquad (3)$$

$$\epsilon_\theta = u/r \qquad (4)$$

$$-\psi = u'\sin\phi + w'\cos\phi \qquad (5)$$

where primes denote differentiation with respect to ϕ.

Constitutive Relations

$$\epsilon_\phi = N_\phi - \nu N_\theta \qquad (6)$$

$$\epsilon_\theta = N_\theta - \nu N_\phi \qquad (7)$$

In these equations the variables have been made dimensionless or scaled according to the following scheme, where bars over quantities indicate the physical or dimensional quantities (see also Fig. 1):

$\bar{N}_\phi = pRN_\phi$ } membrane
$\bar{N}_\theta = pRN_\theta$ } stresses
$\bar{u} = \kappa Ru$ } displacements
$\bar{w} = \kappa Rw$ } displacements
$\bar{\psi} = \kappa\,\psi$ rotation
p = internal pressure
$\bar{\epsilon}_\phi = \kappa\epsilon_\phi$ } membrane
$\bar{\epsilon}_\theta = \kappa\epsilon_\theta$ } strains
h = shell thickness
E = Young's modulus
$\bar{r} = Rr$
$= R(a + \sin\phi)$
ν = Poisson's ratio
$\kappa = pR/Eh$

The linear membrane equations are obtained by setting κ equal to zero in Eqs. (1–5). For reference purposes, the solution of these equations is given here:

$$N_\phi = (a + \tfrac{1}{2}\sin\phi)/(a + \sin\phi) \qquad (8)$$

$$N_\theta = \tfrac{1}{2} \qquad (9)$$

$$u = \tfrac{1}{2}[a(1 - 2\nu) + (1 - \nu)\sin\phi] \qquad (10)$$

$$\psi = a\cos\phi/2\sin\phi\,(a + \sin\phi) \qquad (11)$$

$$w = \tfrac{1}{2}(1 - \nu)\cos\phi - \tfrac{1}{2}\log|\tan(\phi/2)| \mp \text{arc tan}\left[\left(\frac{a-1}{a+1}\right)^{1/2}\frac{1 - \tan(\phi/2)}{1 + \tan(\phi/2)}\right]^{\pm 1} \qquad (12)$$

where the upper sign holds if $\phi > 0$ and the lower sign holds if $\phi < 0$. Obviously, there are discontinuities in w and ψ at $\phi = 0$; nevertheless, it is expected that this solution will be at

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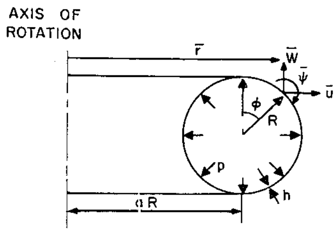

Fig. 1 Geometry and notation.

least approximately correct except near $\phi = 0$. Note that the variables have been scaled in such a way as to be 0(1) except possibly near $\phi = 0$. In particular, the physical strains ($\bar{\epsilon}_\phi$ and $\bar{\epsilon}_\theta$) are of the order of κ, which is assumed to be small compared to unity.

Solution of the Nonlinear Equations

The authors intend to find an approximate solution to the nonlinear equations accurate to within errors of a stated order of magnitude. In the process of solving the equations, certain order-of-magnitude assumptions will be made to be verified later when the solution is obtained. At the outset it is assumed that $\psi \sin\phi$, N_ϕ, and N_θ are 0(1) for all ϕ, and that $\psi = 0(1)$ except for $\phi = 0(\kappa^{1/4})$, where $\psi = 0(\kappa^{-1/4})$. It is assumed also that differentiation increases the order of magnitude of any quantity by a factor $\kappa^{-1/4}$ whenever $\phi = 0(\kappa^{-1/4})$.

The solution is begun by integrating Eq. (1) to obtain the following:

$$r(\sin\phi + \kappa\psi\cos\phi)N_\phi = a\sin\phi + \tfrac{1}{2}\sin^2\phi - a\sin\phi_0 - \tfrac{1}{2}\sin^2\phi_0 + 0(\kappa)$$
$$= a\sin\phi + \tfrac{1}{2}\sin^2\phi - a\phi_0 + 0(\kappa) \quad (13)$$

The constant ϕ_0 appearing in this equation is assumed to be $0(\kappa^{3/4})$. The angle of inclination of the normal of the deformed shell to the axis of the torus is $\phi + \kappa\psi$. Now $\sin(\phi + \kappa\psi) \approx \sin\phi + \kappa\psi\cos\phi$ vanishes when this normal becomes parallel to the axis. The value of ϕ for which this occurs has been denoted by ϕ_0.

Next Eq. (1) is multiplied by $\sin\phi + \kappa\psi\cos\phi$ and Eq. (2) by $\cos\phi - \kappa\psi\sin\phi$ to obtain

$$(1 + \kappa^2\psi^2)(rN_\phi)' + \kappa^2\psi\psi'(rN_\phi) - (\cos\phi - \kappa\psi\sin\phi)N_\theta = 0$$

or

$$(rN_\phi)' = \cos\phi\, N_\theta + 0(\kappa) \quad (14)$$

Now from Eqs. (3–5), it follows that

$$\epsilon_\phi\cos\phi - \psi\sin\phi = (r\epsilon_\theta)' + \tfrac{1}{2}\kappa\psi^2\cos\phi \quad (15)$$

$$\epsilon_\phi\sin\phi + \psi\cos\phi = -w' + \tfrac{1}{2}\kappa\psi^2\sin\phi \quad (16)$$

From Eqs. (6, 7, and 15), it follows that

$$(N_\phi - \nu N_\theta)\cos\phi = \psi\sin\phi + \cos\phi N_\theta + rN_\theta' - \nu(rN_\phi)' + \tfrac{1}{2}\kappa\psi^2\cos\phi \quad (17)$$

which, with the help of Eq. (14), becomes

$$\cos\phi(N_\phi - N_\theta) = \psi\sin\phi + rN_\theta' + 0(\kappa^{1/2}) \quad (18)$$

Equation (16) may be rewritten as follows:

$$w' = -\psi\cos\phi - \sin\phi(N_\phi - \nu N_\theta) + 0(\kappa^{3/4}) \quad (19)$$

At this stage the problem has been reduced to the simultaneous solution of the two equilibrium equations (13) and (14), the compatibility equation (18), and Eq. (19) for the determination of w.

A new independent variable now is introduced, defined by

$$\eta = \phi - \phi_0 \quad (20)$$

and a variable Φ is defined by

$$\phi_0 + \kappa\psi = \Phi\sin\eta \quad (21)$$

As previously noted, $\phi_0 + \kappa\psi$ vanishes for $\phi = \phi_0$ or for $\eta = 0$. According to the order of magnitude assumptions,

$$\sin\eta\ \Phi = 0(\kappa^{3/4}) \qquad \text{for all } \eta$$
$$\Phi = 0(\kappa^{1/2}) \qquad \text{for } \eta = 0(\kappa^{1/4})$$

The following equations can be derived easily:

$$r = a + \sin\eta + \phi_0\cos\eta + 0(\kappa^{3/2})$$

$$\sin\phi + \kappa\psi\cos\phi = \sin\eta(1 + \Phi\cos\eta) + 0(\kappa^{3/2})$$

$$a\sin\phi - a\phi_0 + \tfrac{1}{2}\sin^2\phi = \sin\eta(a + \tfrac{1}{2}\sin\eta) + \phi_0\,[\sin\eta\cos\eta - a(1 - \cos\eta)] + 0(\kappa^{3/2})$$

Now Eq. (13) may be solved for rN_ϕ to obtain

$$rN_\phi = (a + \tfrac{1}{2}\sin\eta)(1 - \cos\eta\,\Phi) + \phi_0\{\cos\eta - a[(1 - \cos\eta)/\sin\eta]\} + 0(\kappa) \quad (22)$$

This step effectively makes the problem a linear one! Differentiation of this equation gives

$$(rN_\phi)' = \tfrac{1}{2}\cos\eta(1 - \cos\eta\,\Phi) - (a + \tfrac{1}{2}\sin\eta) \times \cos\eta\,\Phi' + 0(\kappa^{3/4}) \quad (23)$$

From Eq. (14) and $\cos\phi = \cos\eta + 0(\kappa^{3/4})$, it follows that

$$N_\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos\eta\,\Phi - (a + \tfrac{1}{2}\sin\eta)\,\Phi' + 0(\kappa^{3/4}) \quad (24)$$

Differentiation of this equation gives

$$N_\theta' = -(a + \tfrac{1}{2}\sin\eta)\,\Phi'' - \cos\eta\,\Phi' + 0(\kappa^{1/2}) \quad (25)$$

At this point, N_ϕ and N_θ are expressed (linearly) in terms of Φ and the unknown constant ϕ_0 by Eqs. (22) and (24); one has only to substitute these results into Eq. (18) in order to obtain a differential equation for Φ. The result of the substitution is

$$\alpha_1\alpha_2\Phi'' + (\alpha_1 + \alpha_2)\cos\eta\Phi' - \kappa^{-1}\sin^2\eta\Phi = -(\phi_0/\kappa)\sin\eta - \tfrac{1}{2}a\alpha_1^{-1}\cos\eta + 0(\kappa^{1/2}) \quad (26)$$

where

$$\alpha_1 = a + \sin\eta \qquad \alpha_2 = a + \tfrac{1}{2}\sin\eta$$

The problem has been reduced to the solution of this differential equation. This equation, which contains the large parameter $1/\kappa$ and has a double turning point at $\eta = 0$, can be solved, approximately, by known methods of asymptotic integration as explained in Refs. 3 and 9, for example.

Solution of the Equation for Φ

First introduce a new dependent variable u defined by

$$u = \alpha_1^{1/2}\alpha_2\Phi \quad (27)$$

The differential equation satisfied by u is

$$u'' - \kappa^{-1}qu = -\kappa^{-1}\alpha_1^{-1/2}\phi_0 \times \sin\eta - \tfrac{1}{2}\alpha_1^{-3/2}a\cos\eta + 0(\kappa^{1/2}) \quad (28)$$

where

$$q = (\alpha_1\alpha_2)^{-1}\sin^2\eta \quad (29)$$

A new independent variable is introduced next, defined by

$$\xi = \left(2\int_0^\eta q^{1/2}d\eta\right)^{1/2} \quad (30)$$

This defines ξ as a real, single-valued function of η in the interval $-\pi < \eta < \pi$. See the Appendix for the evaluation of this integral. Finally, another new dependent variable y is introduced, defined by

$$u = \xi^{1/2} q^{-1/4} y \tag{31}$$

The differential equation satisfied by y is

$$\ddot{y} - \kappa^{-1}\xi^2 y = -(\xi^2/q)^{3/4}[\kappa^{-1}\phi_0\alpha_1^{-1/2}\sin\eta + \tfrac{1}{2}a\alpha_1^{-3/2}\cos\eta] + 0(\kappa^{1/2}) = F(\xi) + 0(\kappa^{1/2}) \tag{32}$$

where a dot denotes differentiation with respect to ξ. Now if ξ is not small, the second derivative term on the left-hand side of Eq. (32) can be neglected, and one has approximately

$$y = \kappa F(\xi)/\xi^2 + 0(\kappa^{3/2}) \tag{33}$$

and this can be shown to lead to agreement with the linear membrane solution. However, since $F(0) \neq 0$, one obviously needs something different for small ξ. One must seek a solution of (32) continuous for $\xi = 0$ and asymptotically the same as the expression in (33) as ξ becomes large. Some preliminary manipulations are useful in simplifying the problem. Let (32) be written in the form

$$\ddot{y} - \kappa^{-1}\xi^2 y = F(0) + \xi\dot{F}(0) + \xi^2\rho(\xi) + 0(\kappa^{1/2}) \tag{34}$$

where

$$\rho(\xi) = (1/\xi^2)\,[F(\xi) - F(0) - \xi\dot{F}(0)] \tag{34a}$$

By direct calculation, it can be shown that

$$F(0) = -\tfrac{1}{2}a^{1/4} \tag{35}$$

$$\dot{F}(0) = -a^{3/4}\,[(\phi_0/\kappa) - (3/8a)] \tag{36}$$

Now define functions $T_1(x)$ and $T_2(x)$ such that

$$T_1'' - x^2 T_1 = -1 \qquad T_1 \sim 1/x^2 \quad \text{as } |x| \to \infty \tag{37}$$

$$T_2'' - x^2 T_2 = -x \qquad T_2 \sim 1/x \quad \text{as } |x| \to \infty \tag{38}$$

In terms of the functions T_1, T_2, and ρ, the solution of Eq. (32) is

$$y(\xi) = -\kappa^{1/2}F(0)T_1(\kappa^{-1/4}\xi) - \kappa^{3/4}\dot{F}(0)T_2(\kappa^{-1/4}\xi) - \kappa\rho(\xi) + \epsilon \tag{39}$$

where ϵ is an error term with the following order of magnitude:

$$\epsilon = 0(\kappa) \qquad \text{for } \xi = 0(\kappa^{1/4})$$
$$= 0(\kappa^{3/2}) \qquad \text{for } \xi = 0(1)$$

The differential equations (37) and (38) are solved in the Appendix, and the functions T_1 and T_2 are tabulated in Table 1.

From (39) and previous equations, the result for Φ is

$$\Phi = \phi_0(\sin\eta)^{-1} + \tfrac{1}{2}\kappa a\alpha_1^{-1}\cos\eta(\sin\eta)^{-2} - (\alpha_1\alpha_2^3)^{-1/4}(\xi/\sin\eta)^{1/2}\{\kappa^{1/2}F(0)\,[T_1(\xi\kappa^{-1/4}) - \xi^{-2}\kappa^{1/2}] + \kappa^{3/4}\dot{F}(0)\,[T_2(\xi\kappa^{-1/4}) - \xi^{-1}\kappa^{1/4}]\} + \epsilon' \tag{40}$$

where ϵ' is an error of the same order as ϵ. The membrane stresses N_ϕ and N_θ are given by

$$N_\phi = \alpha_2\alpha_1^{-1}(1 - \cos\eta\,\Phi) + 0(\kappa^{3/4}) \tag{41}$$

$$N_\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos\eta\,\Phi - \alpha_2\,\Phi' + 0(\kappa^{3/4}) \tag{42}$$

In these equations η may be replaced by ϕ with negligible error. For purposes of computing with these formulas, it is convenient to have Φ and Φ' in the following forms:

$$\Phi = -\alpha_1^{-1/4}\alpha_2^{-3/4}\left(\frac{\xi}{\sin\eta}\right)^{1/2}[\kappa^{1/2}F(0)T_1(\xi\kappa^{-1/4}) + \kappa^{3/4}\dot{F}(0)T_2(\xi\kappa^{-1/4})] + 0(\kappa^{3/4})$$

$$\Phi' = -\alpha_1^{-3/4}\alpha_2^{-5/4}\left(\frac{\xi}{\sin\eta}\right)^{-1/2}[\kappa^{1/4}F(0)T_1'(\xi\kappa^{-1/4}) + \kappa^{1/2}\dot{F}(0)T_2'(\xi\kappa^{-1/4})] + \tfrac{1}{8}\alpha_1^{-5/4}\alpha_2^{-7/4}\cos\eta\left(\frac{\xi}{\sin\eta}\right)^{1/2}\left\{3\alpha_1 + 2\alpha_2 + \frac{4\alpha_1\alpha_2}{\sin\eta}\left[1 - \frac{1}{(\alpha_1\alpha_2)^{1/2}\cos\eta}\left(\frac{\sin\eta}{\xi}\right)^2\right]\right\} \times [\kappa^{1/2}F(0)T_1(\xi\kappa^{-1/4}) + \kappa^{3/4}\dot{F}(0)T_2(\xi\kappa^{-1/4})] + 0(\kappa^{3/4})$$

Determination of w and ϕ_0

From Eq. (19) and subsequent equations it follows that

$$w' = \kappa^{-1}(\Phi\sin\eta - \phi_0)\cos\eta - \sin\eta\,(\alpha_2\alpha_1^{-1} - \nu/2) + 0(\kappa^{1/2}) \tag{43}$$

From (40) and (43) and after considerable algebra, one obtains

$$w(\eta) - w(0) = -a^{3/4}\left(\frac{\phi_0}{\kappa} - \frac{3}{8a}\right)\int_0^\xi \frac{\xi\cos\eta}{\kappa^{1/4}}\left(\frac{\alpha_1}{\alpha_2}\right)^{1/4}\left(\frac{\xi}{\sin\eta}\right)^{1/2} \times \left\{T_2\left(\frac{\xi}{\kappa^{1/4}}\right) - \frac{\kappa^{1/4}}{\xi}\right\}d\xi - \frac{1}{2}\int_0^\xi \frac{\xi}{\kappa^{1/2}} \times \left[\cos\eta\left(\frac{a\alpha_1}{\alpha_2}\right)^{1/4}\left(\frac{\xi}{\sin\eta}\right)^{1/2} - 1\right]\left\{T_1\left(\frac{\xi}{\kappa^{1/4}}\right) - \frac{\kappa^{1/2}}{\xi^2}\right\}d\xi - \frac{1}{2}\int_0^\xi \frac{\xi}{\kappa^{1/2}}T_1\left(\frac{\xi}{\kappa^{1/4}}\right)d\xi + \frac{1}{2}\int_0^\xi\left[\frac{1}{\xi} - \frac{a\xi\cos^2\eta}{\sin^2\eta}\left(\frac{\alpha_2}{\alpha_1}\right)^{1/2} - 2\xi(\alpha_1\alpha_2)^{1/2}\left(\frac{\alpha_2}{\alpha_1} - \frac{\nu}{2}\right)\right]d\xi + 0(\kappa^{1/2}) \tag{44}$$

In the first two integrals, the asymptotic behavior of the contents of the braces is such that the rest of the integrand can be replaced by its approximation for small ξ with negligible error. If one lets $\xi = \kappa^{1/4}x$, the first integral becomes

$$\left(\frac{\kappa}{a}\right)^{1/4} I_2(\xi\kappa^{-1/4}) \text{ where } I_2(x) = \int_0^x [xT_2(x) - 1]dx$$

The second integral turns out to be $0(\kappa^{1/2})$ and hence is negligible. The third integral becomes

$$I_1(\xi\kappa^{-1/4}) \text{ where } I_1(x) = \int_0^x xT_1(x)dx$$

Numerical values of I_1 and I_2, which are even and odd functions, respectively, are given in Table 1. With the help of the formula

$$d\xi/d\eta = \xi^{-1}(\alpha_1\alpha_2)^{-1/2}\sin\eta$$

the fourth integral can be integrated exactly. The complete expression for w is as follows:

$$w(\eta) - w(0) = \kappa^{1/4}a^{1/2}(3/8a - \kappa^{-1}\phi_0)I_2(\xi\kappa^{-1/4}) - \tfrac{1}{2}I_1(\xi\kappa^{-1/4}) + \tfrac{1}{2}\log[a^{1/2}\xi(\sin\eta)^{-1}\cos^2(\eta/2)] - \tfrac{1}{2}(1-\nu)(1-\cos\eta) + (a^2-1)^{-1/2}\arctan\{(a^2-1)^{1/2}[1 + a\cot(\eta/2)]^{-1}\} + 0(\kappa^{1/2}) \tag{45}$$

Since $w(\pi/2) = 0$ from symmetry, the constant $w(0)$ can be determined by setting $\eta = \pi/2$ in the foregoing formula.

Table 1

X	T_1	T_2	T_1'	T_2'	I_1	I_2	X	T_1	T_2	T_1'	T_2'	I_1	I_2
0	1.3110	0	0	0.599	0	0	5.1	0.0388	0.1967	−0.016	−0.039	2.482	−1.067
0.1	1.3060	0.0597	−0.100	0.594	0.007	−0.100	5.2	0.0373	0.1929	−0.015	−0.038	2.501	−1.067
0.2	1.2912	0.1185	−0.197	0.579	0.026	−0.198	5.3	0.0359	0.1892	−0.014	−0.036	2.520	−1.006
0.3	1.2669	0.1753	−0.288	0.555	0.058	−0.295	5.4	0.0346	0.1856	−0.013	−0.035	2.539	−1.066
0.4	1.2338	0.2293	−0.373	0.523	0.102	−0.388	5.5	0.0333	0.1822	−0.012	−0.033	2.558	−1.066
0.5	1.1926	0.2796	−0.448	0.483	0.156	−0.476	5.6	0.0321	0.1790	−0.012	−0.032	2.576	−1.066
0.6	1.1444	0.3257	−0.513	0.437	0.221	−0.559	5.7	0.0310	0.1758	−0.011	−0.031	2.594	−1.065
0.7	1.0904	0.3669	−0.566	0.387	0.293	−0.637	5.8	0.0299	0.1727	−0.010	−0.030	2.611	−1.065
0.8	1.0317	0.4030	−0.606	0.334	0.373	−0.708	5.9	0.0289	0.1698	−0.010	−0.029	2.628	−1.065
0.9	0.9696	0.4336	−0.634	0.279	0.458	−0.772	6.0	0.0279	0.1669	−0.009	−0.028	2.645	−1.065
1.0	0.9054	0.4588	−0.649	0.224	0.547	−0.830	6.1	0.0270	0.1642	−0.009	−0.027	2.662	−1.065
1.1	0.8402	0.4785	−0.653	0.171	0.638	−0.880	6.2	0.0261	0.1615	−0.009	−0.026	2.678	−1.065
1.2	0.7752	0.4931	−0.646	0.121	0.731	−0.925	6.3	0.0253	0.1589	−0.008	−0.025	2.694	−1.065
1.3	0.7113	0.5028	−0.630	0.074	0.824	−0.962	6.4	0.0245	0.1564	−0.008	−0.025	2.710	−1.064
1.4	0.6494	0.5080	−0.606	0.031	0.916	−0.994	6.5	0.0238	0.1540	−0.007	−0.024	2.726	−1.064
1.5	0.5902	0.5091	−0.676	−0.007	1.006	−1.020	6.6	0.0230	0.1517	−0.007	−0.023	2.741	−1.064
1.6	0.5344	0.5067	−0.541	−0.040	1.093	−1.041	6.7	0.0223	0.1494	−0.007	−0.022	2.756	−1.064
1.7	0.4821	0.5012	−0.503	−0.068	1.176	−1.058	6.8	0.0217	1.1472	−0.006	−0.022	2.771	−1.064
1.8	0.4338	0.4933	−0.463	−0.090	1.257	−1.071	6.9	0.0211	0.1451	−0.006	−0.021	2.785	−1.064
1.9	0.3896	0.4833	−0.422	−0.108	1.333	−1.081	7.0	0.0205	1.1430	−0.006	−0.021	2.800	−1.064
2.0	0.3494	0.4718	−0.382	−0.122	1.405	−1.088	7.1	0.0199	0.1410	−0.006	−0.021	2.814	−1.064
2.1	0.3132	0.4591	−0.343	−0.131	1.472	−1.092	7.2	0.0193	0.1390	−0.005	−0.019	2.828	−1.064
2.2	0.2807	0.4457	−0.306	−0.137	1.536	−1.095	7.3	0.0188	0.1371	−0.005	−0.019	2.842	−1.064
2.3	0.2518	0.4318	−0.271	−0.140	1.596	−1.096	7.4	0.0183	0.1352	−0.005	−0.018	2.856	−1.063
2.4	0.2264	0.4178	−0.239	−0.140	1.652	−1.096	7.5	0.0178	0.1334	−0.005	−0.018	2.869	−1.063
2.5	0.2039	0.4039	−0.211	−0.139	1.705	−1.096	7.6	0.0173	0.1316	−0.005	−0.017	2.882	−1.063
2.6	0.1842	0.3902	−0.185	−0.135	1.754	−1.095	7.7	0.0169	0.1299	−0.004	−0.017	2.895	−1.063
2.7	0.1669	0.3768	−0.162	−0.131	1.801	−1.093	7.8	0.0165	0.1283	−0.004	−0.016	2.908	−1.063
2.8	0.1517	0.3640	−0.141	−0.126	1.844	−1.091	7.9	0.0161	0.1266	−0.004	−0.016	2.921	−1.063
2.9	0.1386	0.3517	−0.123	−0.120	1.886	−1.089	8.0	0.0156	0.1251	−0.004	−0.016	2.934	−1.063
3.0	0.1270	0.3399	−0.108	−0.115	1.925	−1.087	8.1	0.0153	0.1235	−0.004	−0.015	2.946	−1.063
3.1	0.1169	0.3287	−0.095	−0.109	1.962	−1.085	8.2	0.0149	0.1220	−0.004	−0.015	2.958	−1.063
3.2	0.1080	0.3182	−0.083	−0.103	1.997	−1.083	8.3	0.0143	0.1205	−0.004	−0.015	2.971	−1.063
3.3	0.1001	0.3082	−0.074	−0.097	2.031	−1.082	8.4	0.0142	0.1191	−0.003	−0.014	2.982	−1.063
3.4	0.0932	0.2987	−0.065	−0.092	2.063	−1.080	8.5	0.0139	0.1177	−0.003	−0.014	2.994	−1.063
3.5	0.0870	0.2898	−0.058	−0.087	2.094	−1.079	8.6	0.0135	0.1163	−0.003	−0.014	3.006	−1.063
3.6	0.0815	0.2814	−0.052	−0.082	2.124	−1.077	8.7	0.0132	0.1150	−0.003	−0.013	3.018	−1.063
3.7	0.0766	0.2735	−0.047	−0.077	2.153	−1.076	8.8	0.0129	0.1137	−0.003	−0.013	3.029	−1.063
3.8	0.0722	0.2660	−0.042	−0.073	2.181	−1.075	8.9	0.0126	0.1124	−0.003	−0.013	3.040	−1.063
3.9	0.0681	0.2589	−0.038	−0.069	2.208	−1.074	9.0	0.0124	0.1112	−0.003	−0.012	3.052	−1.063
4.0	0.0645	0.2522	−0.035	−0.065	2.234	−1.073	9.1	0.0121	0.1099	−0.003	−0.012	3.063	−1.063
4.1	0.0611	0.2458	−0.032	−0.062	2.260	−1.072	9.2	0.0118	0.1087	−0.003	−0.012	3.074	−1.063
4.2	0.0581	0.2398	−0.029	−0.059	2.284	−1.071	9.3	0.0116	0.1076	−0.002	−0.012	3.084	−1.063
4.3	0.0552	0.2341	−0.027	−0.056	2.308	−1.071	9.4	0.0113	0.1064	−0.002	−0.011	3.095	−1.063
4.4	0.0526	0.2286	−0.025	−0.053	2.332	−1.070	9.5	0.0111	0.1053	−0.002	−0.011	3.106	−1.063
4.5	0.0502	0.2234	−0.023	−0.051	2.355	−1.069	9.6	0.0109	0.1042	−0.002	−0.011	3.116	−1.063
4.6	0.0480	0.2184	−0.022	−0.048	2.377	−1.069	9.7	0.0106	0.1031	−0.002	−0.011	3.127	−1.063
4.7	0.0459	0.2137	−0.020	−0.046	2.399	−1.068	9.8	0.0104	0.1021	−0.002	−0.010	3.137	−1.063
4.8	0.0439	0.2092	−0.019	−0.044	2.420	−1.068	9.9	0.0102	0.1010	−0.002	−0.010	3.147	−1.062
4.9	0.0421	0.2048	−0.018	−0.042	2.441	−1.068	10.0	0.0100	0.1000	−0.002	−0.010	3.157	−1.062
5.0	0.0404	0.2007	−0.017	−0.041	2.462	−1.067							

Here also, replacing η by ϕ results in a negligible error. The unknown constant ϕ_0 can be determined from the other symmetry condition $w(-\pi/2) = 0$, or equivalently from the equation

$$w(\pi/2) - w(-\pi/2) = 0 \tag{46}$$

From Eq. (45) this gives

$$-\kappa^{1/4}a^{1/2}(\kappa^{-1}\phi_0 - 3/8a)\{I_2(\xi_+\kappa^{-1/4}) - I_2(\xi_-\kappa^{-1/4})\} + \tfrac{1}{2}\log|\xi_+/\xi_-| - \tfrac{1}{2}\{I_1(\xi_+\kappa^{-1/4}) - I_1(\xi_-\kappa^{-1/4})\} + (a^2 - 1)\left\{\tan^{-1}\left(\frac{a-1}{a+1}\right)^{1/2} - \tan^{-1}\left(\frac{a+1}{a-1}\right)^{1/2}\right\} = 0(\kappa^{1/2}) \tag{47}$$

where $\xi = \xi_\pm$ when $\eta = \pm\pi/2$. From the asymptotic formulas for T_1 and T_2 given in the Appendix, it follows that

$$I_1(x) \sim I_1(10) + \log|x/10| + 0(x^{-4}) \tag{48}$$

and that

$$I_2(x) \sim \int_0^\infty (xT_2 - 1)\, dx + 0(x^{-4}) \tag{49}$$

for x positive (I_2 is odd). The infinite integral is evaluated in the Appendix. The result is

$$I_2(\infty) = -2^{-1/2}[\Gamma(\tfrac{3}{4})]^2 \tag{50}$$

Now $\xi_\pm\kappa^{-1/4}$ are sufficiently large numbers so that these asymptotic formulas can be used in Eq. (47) with no loss in accuracy. Solving for ϕ_0 gives the result

$$\phi_0 = -\tfrac{1}{2}\pi\kappa^{3/4}[2a(a^2 - 1)]^{-1/2}\,[\Gamma(\tfrac{3}{4})]^{-2} + 3\kappa/8a + 0(\kappa^{5/4}) \tag{51}$$

This calculation confirms the assumption that $\phi_0 = 0(\kappa^{3/4})$.

Concluding Remarks

The solution to the problem of a toroidal shell under internal pressure has exploited methods of asymptotic integration of differential equations based on the assumption that κ is a small parameter. There is, however, a restriction on the validity of the present solution which has not been discussed in detail. According to Reissner,[5] bending effects are not negligible unless $\kappa \gg (h/R)^{4/3}$. The present theory is valid if the shell is sufficiently thin.

The stress resultants and the deformed shape for $a = 1.5$, $\kappa = 0.002$, and $\nu = 0.3$ are presented in Figs. 2–4. For this particular geometry and load factor, the correction to the linear N_ϕ is negligible, but it is approximately 18% for the linear N_θ. Also, for this geometry and load factor, the results for N_ϕ and N_θ from this investigation compare well with those computed by Jordan.[4]

Appendix

Solution of Differential Equations

The differential equations to be solved are

$$T_1'' - x^2 T_1 = -1 \tag{A1}$$

$$T_2'' - x^2 T_2 = -x \tag{A2}$$

subject to the conditions

$$\begin{matrix} T_1 \sim x^{-2} \\ T_2 \sim x^{-1} \end{matrix} \qquad \text{as } |x| \to \infty \tag{A3}$$

The solution of the second equation will be given in some detail but only the results for the first.

Conditions are such that the Mellin transform of T_2 exists. Write T_2 as an inverse Mellin transform:

$$T_2(x) = \frac{1}{2\pi i} \int_\gamma f(s)\, x^s ds \tag{A4}$$

where γ is an infinite straight line path in the complex plane parallel to the imaginary axis and in the upward sense. Substitution of (A4) into (A1) gives

$$\frac{1}{2\pi i} \int_\gamma s(s-1)\, f(s)\, x^{s-2} ds - \frac{1}{2\pi i} \times \int_{\gamma+4} f(s-4) x^{s-2} ds = -x \tag{A5}$$

Let $f(s)$ satisfy the following difference equation:

$$f(s-4) = s(s-1) f(s) \tag{A6}$$

Then (A5) can be written as

$$\frac{1}{2\pi i} \oint_c s(s-1) f(s) x^{s-2} ds = x \tag{A7}$$

where c is the combined path, closed at infinity, taken in the usual positive sense. The general solution of Eq. (A6) is

$$f(s) = \frac{P(s)}{2^s \Gamma[(s+4)/4]\, \Gamma[(s+3)/4]} \tag{A8}$$

where $P(s) = P(s-4)$ is a periodic function to be determined. Equation (A7) will be satisfied if the path c encloses the point $s = 3$, $s(s-1)f(s)$ has a simple pole of residue 1 at $s = 3$ and no other singularities inside the path c, and if the integral (A7) converges. The condition (A3) will be met if the path γ lies to the left of the imaginary axis.

All of these conditions will be met if

$$P(s) = \pi^{3/2}\Gamma(\tfrac{3}{4})/4 \cos(\pi\, s/2) \tag{A9}$$

and if γ passes through $s = -\frac{1}{2}$. The result is

$$T_2(x) = \frac{\pi^{3/2}\Gamma(\frac{3}{4})}{8\pi i} \times \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{x^s ds}{2^s \Gamma[(s+4)/4]\, \Gamma[(s+3)/4] \cos(\pi s/2)} \tag{A10}$$

The convergent power series for T_2 can be obtained by summing residues to the right of the path. The result is

$$T_2(x) = \frac{[\Gamma(\frac{3}{4})]^2}{(2\pi)^{1/2}} \sum_0^\infty \frac{x^{4n+1}}{1\cdot5\cdot9\cdots(4n+1)n!2^{2n}} - \sum_0^\infty \frac{x^{4n+3}}{3\cdot7\cdot11\cdots(4n+3)\cdot1\cdot3\cdot5\cdots(2n+1)2^{n+1}} \tag{A11}$$

It is possible to prove rigorously that the asymptotic series for T_2 is obtained by summing residues to the left of the path γ. The result is

$$T_2(x) \sim \frac{1}{x} + \sum_1^\infty 1\cdot5\cdot9\cdots(4n-3)\cdot1\cdot3\cdot5\cdots(2n-1)\frac{2^n}{x^{4n+1}} \tag{A12}$$

The results for T_1 are

$$T_1(x) = \frac{[\Gamma(\frac{1}{4})]^2}{4(2\pi)^{1/2}}\left[1 + \sum_0^\infty \frac{x^{4n}}{n!\,3\cdot7\cdot11\cdots(4n-1)2^{2n}}\right] - \sum_0^\infty \frac{x^{4n+2}}{1\cdot3\cdot5\cdots(2n+1)\cdot1\cdot5\cdot9\cdots(4n+1)2^{n+1}} \tag{A13}$$

$$T_1(x) \sim \frac{1}{x^2} + \sum_1^\infty 1\cdot3\cdot5\cdots(2n-1)\cdot3\cdot7\cdot11\cdots(4n-1) \times \frac{2^n}{x^{4n+2}} \tag{A14}$$

Series for T_1' and T_2' are obtained easily from the foregoing formulas. The functions I_1 and I_2 were obtained by numerical integration.

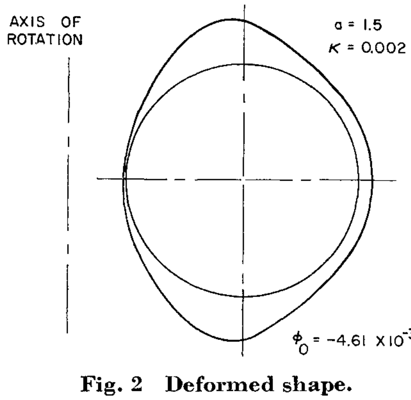

Fig. 2 Deformed shape.

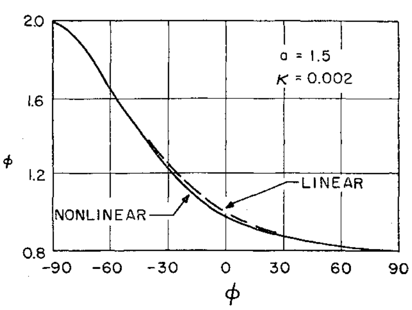

Fig. 3 Stress resultant $N\phi$.

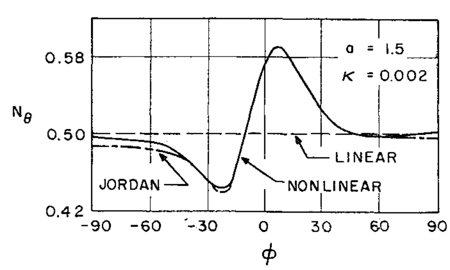

Fig. 4 Stress resultant $N\theta$.

Evaluation of the Integral $I_2(\infty)$

One has

$$I_2(\infty) = \int_0^\infty (xT_2 - 1)dx = -1 + \int_0^1 xT_2dx + \int_1^\infty (xT_2 - 1)dx \quad \text{(A15)}$$

From Eq. (A4),

$$T_2 = \frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} f(s)x^s ds \quad \text{(A16)}$$

From this one obtains, by integration under the integral signs, the following:

$$\int_0^1 xT_2dx = \frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} f(s)\frac{ds}{s+2} \quad \text{(A17)}$$

since

$$x^{s+2}\big|_0^1 = 1 \qquad \text{when } R\{s+2\} = \tfrac{3}{2} > 0$$

If in the integral (A16) for T_2 one moves the path to the left two units and accounts for the pole at $s = -1$, one obtains

$$T_2 = \frac{1}{x} + \frac{1}{2\pi i}\int_{-5/2-i\infty}^{-5/2+i\infty} f(s)x^s ds \quad \text{(A18)}$$

from which it follows that

$$\int_1^\infty (xT_2 - 1)dx = -\frac{1}{2\pi i}\int_{-5/2-i\infty}^{-5/2+i\infty} f(s)\frac{ds}{s+2} \quad \text{(A19)}$$

since

$$x^{s+2}\Big|_1^\infty = -1 \qquad \text{when } R\{s+2\} = -\tfrac{1}{2} < 0$$

Combining these results gives

$$\int_0^\infty (xT_2 - 1)dx = -1 + \frac{1}{2\pi i}\oint_c f(s)\frac{ds}{s+2} \quad \text{(A20)}$$

where c is the resultant of the paths in (A17) and (A19) taken in the positive sense. The integral on the right of (A20) is evaluated easily to yield

$$I_2(\infty) - 2^{-1/2}[\Gamma(\tfrac{3}{4})]^2 \quad \text{(A21)}$$

ξ in Terms of η

By definition,

$$\frac{1}{2}\xi^2 = \int_0^\eta q^{1/2}\, d\eta \quad \text{(A22)}$$

where

$$q = (\alpha_1\alpha_2)^{-1}\sin^2\eta = \rho^2/(1 - 1/16\rho^2) \quad \text{(A23)}$$

where

$$\rho = \sin\eta/(a + \tfrac{3}{4}\sin\eta) \quad \text{(A24)}$$

From this, one obtains the following series expansion of $q^{1/2}$:

$$q^{1/2} = \rho + \tfrac{1}{32}\rho^2 + \tfrac{3}{2048}\rho^3 + \ldots \quad \text{(A25)}$$

which converges very rapidly. By retaining the first two terms, one obtains

$$\frac{1}{2}\xi^2 = \frac{38}{27}\eta - \frac{(38 - 77\lambda^2 + 42\lambda^4)}{27(1-\lambda^2)^{5/2}}\left[\arcsin\frac{\lambda + \sin\eta}{1 + \lambda\sin\eta} - \arcsin\lambda\right] - \frac{\lambda}{27(1-\lambda^2)}\left[\frac{\cos\eta}{(1+\lambda\sin\eta)^2} - 1\right] + \frac{\lambda(1-2\lambda^2)}{9(1-\lambda^2)}\left[\frac{\cos\eta}{1+\lambda\sin\eta} - 1\right] \quad \text{(A26)}$$

where

$$\lambda = 3/4a \quad \text{(A27)}$$

For small values of η the following approximation holds:

$$\xi \approx a^{-1/2}(\eta - \eta^2/4a) \quad \text{(A28)}$$

It is noted that (22) could be evaluated analytically in terms of elliptic integrals.

References

[1] Timoshenko, S., *Theory of Plates and Shells* (McGraw-Hill Book Co., Inc., New York, 1940), Chap. X.

[2] Clark, R. A., "On the theory of thin elastic toroial shells," J. Math. Phys. **29**, 146–178 (1950).

[3] Tumarkin, S. A., "Asymptotic solution of a linear nonhomogenous second order differential equation with a transition point and its application to the computations of toroidal shells and propeller blades," Appl. Math. Mech. **23(2)**, 1549–1565 (1959).

[4] Jordan, P. F., "Stresses and deformations of the thin-walled pressurized torus," J. Aerospace Sci. **29**, 213–225 (1962).

[5] Reissner, E., "On stresses and deformations in toroidal shells of circular cross-section which are acted upon by uniform normal pressure," Quart. Appl. Math. (to be published).

[6] Reissner, E., "On axisymmetrical deformations of thin shells of revolution," Proc. Symp. Appl. Math. **3**, 27–52 (1950).

[7] Simmonds, J. G., "The general equations of equilibrium of rotationally symmetric membranes and some static solutions of uniform centrifugal loading," NASA TN D-816 (May 1961).

[8] Sanders, J. L., Jr., "Nonlinear theories for thin shells," Quart. Appl. Math. **21**, 21–36 (1963).

[9] Erdelyi, A., *Asymptotic Expansions* (Dover Publications Inc., New York, 1956).